Lagrange's Interpolation Method

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Overview

One of the basic ideas in Mathematics is that of a function and most useful tool of numerical analysis is **interpolation**.

According to Thiele (a numerical analyst), "**Interpolation is the art of reading between the lines of the table.**"

Broadly speaking, interpolation is the problem of obtaining the value of a function for any given functional information about it.

Interpolation technique is used in various disciplines like economics, business, population studies, price determination etc. It is used to fill in the gaps in the statistical data for the sake of continuity of information.

Overview

The concept of interpolation is the selection of a function $p(x)$ from a given class of functions in such a way that the graph of

$$y = p(x)$$

passes through a finite set of given data points. The function $p(x)$ is known as the **interpolating function** or **smoothing function**.

If $p(x)$ is a polynomial, then it is called the **interpolating polynomial** and the process is called the **polynomial interpolation**.

Similarly, if $p(x)$ is a finite trigonometric series, **we have trigonometric interpolation. But we restrict the interpolating function $p(x)$ to being a polynomial.**

Overview

The study of interpolation is based on the calculus of finite differences.

Polynomial interpolation theory has a number of important uses. Its primary uses is to furnish some mathematical tools that are used in developing methods in the areas of approximation theory, numerical integration, and the numerical solution of differential equations.

We discuss Lagrange's formula and error bounds in **two** lectures.

Lagrange Interpolating Polynomial

To generalize the concept of linear interpolation, consider the construction of a polynomial of degree at most n that passes through the $n+1$ points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$.

In this case we need to construct, for each $k = 0, 1, 2, \ldots, n$, a function $L_k(x)$ (called **Lagrange basis**, also called the nth **Lagrange interpolating polynomial**) with the property that $L_k(x_i) = \begin{cases} 0 & \text{when } i \neq k \\ 1 & \text{when } i = k \end{cases}$ hence

$$L_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}.$$

Lagrange Interpolating Polynomial

The interpolating polynomial is easily described once the form of L_k is known, by the following theorem.

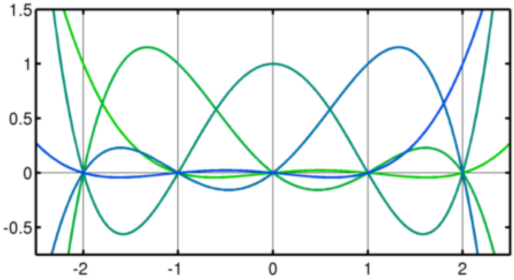
Theorem

If $n+1$ points $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ are given, then a unique polynomial $p_n(x)$ of degree at most n exists with $f(x_k) = p_n(x_k)$ for each $k = 0, 1, \dots, n$. This polynomial is given by

$$p_n(x) = \sum_{k=0}^{n} f(x_k) L_k(x).$$

Graphs of Lagrange Interpolating Polynomials

Given 5 points $(x_0, y_0), (x_1, y_1), \ldots, (x_4, y_4)$, a sketch of the graph of a typical L_k is shown in the following figure.



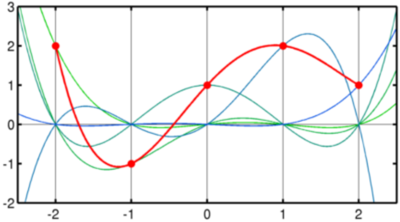
Note how each basis polynomial has a value of 1 for $x = x_k$ $(0 \leq k \leq 4)$, and a value of 0 at all other locations.

Example

Simply multiplying each basis with the corresponding sample value, and adding them all up yields the interpolating polynomial

$$p(x) = \sum_{k=0}^{4} f(x_k) L_k(x).$$

The 5 weighted polynomials are $L_k(x)f(x_k)$ $(0 \leq k \leq 4)$ and their sum (red line) is the interpolating polynomial $p(x)$ (red line) which is shown in the following figure.



How to calculate error bound?

The next step is to calculate a remainder term or bound for the error involved in approximating a function by an interpolating polynomial. This is done in the following theorem.

Theorem (An Important Result for Error Formula)

Suppose $x_0, x_1, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then, for each $x \in [a, b]$, a number $\xi(x)$ (generally unknown) in (a, b) exists with

$$f(x) = p(x) + \frac{(x - x_0)(x - x_1)\cdots(x - x_n)}{(n+1)!} f^{(n+1)}(\xi(x)),$$

where $p(x)$ is the interpolating polynomial given by $p(x) = \sum_{k=0}^{n} f(x_k) L_k(x)$.

The above formula is also called **'Lagrange Error Formula'**.

Error Analysis

The error formula is an important theoretical result because Lagrange polynomials are used extensively for deriving numerical differentiation and integration methods.

Error bounds for these techniques are obtained from the "Lagrange error formula".

Note that the error for the Lagrange polynomial is quite similar to that for the Taylor polynomial.

Comparison of Error Bounds in Taylor and Lagrange Polynomials

The nth **Taylor polynomial** about x_0 concentrates all the known information at x_0 and has an error term of the form

$$\frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi).$$

The **Lagrange polynomial** of degree n uses information at the distinct numbers $x_0, x_1, \ldots, x_n$ and, in place of $(x-x_0)^{n+1}$, its error formula uses a product of the $n+1$ terms $(x-x_0)(x-x_1)\cdots(x-x_n)$

$$\frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(n+1)!} f^{(n+1)}(\xi).$$

Double Interpolation

We have so for derived interpolation formulae to approximate a function of a single variable.

In case of functions of two variables, we interpolate with respect to the first variable keeping the other variable constant. Then interpolate with respect to the second variable.

Similarly, we can extend the said procedure for functions of three variables.

Inverse Interpolation

We have been finding the value of y corresponding to a certain value of x from a given set of values of x and y.

On the other hand, the process of estimating the value of x for a value of y is called **inverse interpolation**. When the values of y are unequally spaced, Lagrange's method is used and when the values of y are equally spaced, the following iterative method is used.

In the procedure, x is assumed to be expressible as a polynomial in y.

Newton's forward interpolation formula is

$$p_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots . \quad (1)$$

From (1) we get

$$p = \frac{1}{\Delta y_0}\left\{ y_p - y_0 - \frac{p(p-1)}{2!}\Delta^2 y_0 - \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 - \cdots \right\}.$$

Neglecting the second and higher differences, we obtain the **first approximation** to p as

$$p_1 = \frac{y_p - y_0}{\Delta y_0}.$$

To find the **second approximation**, retaining the term with second differences in (1) and replacing p by p_1, we get

$$p_2 = \frac{1}{\Delta y_0}\left\{ y_p - y_0 - \frac{p_1(p_1-1)}{2!}\Delta^2 y_0 \right\}.$$

Iterative Method

To find the **third approximation**, retaining the term with third differences in (1) and replacing p by p_2, we get

$$p_3 = \frac{1}{\Delta y_0}\left\{y_p - y_0 - \frac{p_2(p_2-1)}{2!}\Delta^2 y_0 - \frac{p_2(p_2-1)(p_2-2)}{3!}\Delta^3 y_0\right\}$$

and so on. This process is continued till two successive approximations of p agree with each other.

This technique can equally well be applied by any other interpolation formula. This method is a powerful iterative procedure for finding the roots of an equation to a good degree of accuracy.

We shall discuss later some more formulae for finding roots of an equation.

Exercises

1. *Find the polynomial* $f(x)$ *by using Lagrange's formula and hence find* $f(3)$ *for*

x	0	1	2	3
$y = f(x)$	2	3	12	147

2. *A curve passes through the points* $(0, 18), (1, 10), (3, -18)$ *and* $(6, 90)$. *Find the slope of the curve at* $x = 2$.
3. *Using Lagrange's formula, express the function*

$$\frac{3x^2 + x + 1}{(x-1)(x-2)(x-3)}$$

as a sum of partial fractions.

Exercises

4. *Find the missing term in the following table using interpolation.*

x	0	1	2	3	4
$y = f(x)$	1	3	9	-	81

5. *Find the distance moved by a particle and its acceleration at the end of 4 seconds, if the time verses velocity data is as follows.*

t	0	1	3	4
v	21	15	12	10

6. *Using Lagrange's formula prove that*

$$y_0 = \frac{y_1 + y_{-1}}{2} - \frac{1}{8}\left\{\frac{1}{2}(y_3 - y_1) - \frac{1}{2}(y_{-1} - y_{-3})\right\}.$$

[Hint : Here $x_0 = -3, x_1 = -1, x_2 = 1, x_2 = 3.$*]*

Exercises

7. *Given*

$$\begin{aligned} \log_{10} 654 &= 2.8156, \quad \log_{10} 658 = 2.8182, \\ \log_{10} 659 &= 2.8189, \quad \log_{10} 661 = 2.8202 \end{aligned}$$

find by using Lagrange's formula, the value of $\log_{10} 656$.

8. *The following table gives the viscosity of an oil as a function of temperature. Use Lagrange's formula to find viscosity of oil at a temperature of* 140°.

Temperature	110°	130°	160°	190°
Viscosity	*10.8*	*8.1*	*5.5*	*4.8*

9. *Given* $u_1 = 40, u_3 = 45, u_5 = 54$, *find* u_2 *and* u_4.

Exercises

10. *Given* $y_0 = 3, y_1 = 12, y_2 = 81, y_3 = 200, y_4 = 100, y_5 = 8$, *without forming the difference table, find* $\Delta^5 y_0$.

11. *From the data given below, find the number of students whose weight is between* 60 *and* 70.

Weight	*0-40*	*40-60*	*60-80*	*80-100*	*100-120*
No. of Students	*250*	*120*	*100*	*70*	*50*

12. *The values of* $U(x)$ *are known at* a, b, c. *Show that maximum or minimum of Lagrange's interpolation formula is attained at*

$$x = \frac{\sum U_a(b^2 - c^2)}{2\sum U_a(b - c)}.$$

13. *By iterative method, tabulate* $y = x^3$ *for* $x = 2, 3, 4, 5$ *and calculate the cube root of* 10 *correct to* 3 *decimal places.*

Exercises

14. *The following values of $y = f(x)$ are given*

x	10	15	20
y	1754	2648	3564

Find the value of x for $y = 3000$ by iterative method.

15. *Using inverse interpolation, find the real root of the equation $x^3 + x - 3 = 0$ which is close to 1.2.*

Exercises

16. *Solve the equation* $x = 10\log x$, *by iterative method, given that*

x	1.35	1.36	1.37	1.38
$\log x$	0.1303	0.1355	0.1367	0.1392

17. *Apply Lagrange's method, to find the value of* x *when* $f(x) = 15$ *from the given data.*

x	5	6	9	11
$f(x)$	12	13	14	16

18. *The equation* $x^3 - 15x + 4$ *has a root close to* 0.3, *obtain this root upto* 4 *decimal places using inverse interpolation.*

References

1. **Richard L. Burden and J. Douglas Faires**, *Numerical Analysis - Theory and Applications*, Cengage Learning, Singapore.
2. **Kendall E. Atkinson**, *An Introduction to Numerical Analysis*, Wiley India.
3. **David Kincaid and Ward Cheney**, *Numerical Analysis - Mathematics of Scientific Computing*, American Mathematical Society, Providence, Rhode Island.
4. **S.S. Sastry**, *Introductory Methods of Numerical Analysis*, Fourth Edition, Prentice-Hall, India.